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On account of the great simplicity and utility of the method, and of its almost universal omission from texts in English, some readers of the MONTHLY will perhaps be glad to find a portion of General Bixby's pamphlet quoted at length.

The note by Professor Underwood completes the author's previous work (*1921*, 374–376), and bears some relation to his reply to Question 36 (*1922*, 255).

Professor Mathews proves directly a theorem regarding concyclic sets of points on an equilateral hyperbola which is the inverse of a theorem already known on a bicircular quartic.[1] If, in the notation of the paper, the points $\{B\}$ and $\{C\}$ are kept fixed, the sixteen circles AB_iC_jD through a point A define sixteen transformations of A into D. The question raised at the end of the paper is then how to determine the eight points $\{B\}$ and $\{C\}$ so that the transformations generate a finite group.

I. GRAPHICAL SOLUTION OF NUMERICAL EQUATIONS.[2]

By W. H. BIXBY, Washington, D. C.

I was much interested in the article of Professor A. L. Candy on mechanical solution of equations of the nth degree in a single variable.[3] Theoretically, the method is excellent; but practically there will be much difficulty in using it. It will be difficult to determine or secure the exactitude of the right-angles concerned, and the necessary length of the main bar may be awkward. The most perfect and practical mechanism for such equations is the multiple weighing machine, which has several graduated bars and sliding weights, the final weight indicating a root at each place where the machine balances. It would be costly, but no more so than the modern refined planimeters.

From my own personal experience in such matters, the method of Mr. Lill, Austrian engineer, developed by him about 1867 and exhibited by him at the Vienna World Exposition a little later, is the best graphical method yet developed, and far easier, quicker, and more exact, than any other graphical or mechanical method. I read of this about 1878, and published it in 1879 by a privately printed pamphlet.[4] At that date I had not seen Lill's 1867 printed article.[5] A few months ago I found that Luigi Cremona had also described Lill's method and made it public to English readers in 1888.[6] My pamphlet failed to attract much attention. A few engineers showed some interest in it; but apparently there were only a very few who at that time had any occasion for solving such equations in their daily practice.

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[1] R. Lachlan, "On a theorem relating to bicircular quartics and twisted cubics," *Proceedings of the London Mathematical Society*, vol. 21, 1890, pp. 274–280; J. L. Coolidge, *A Treatise on the Circle and the Sphere*, Oxford, 1915, p. 217.

[2] Extract from a letter to the editors.

[3] A. L. Candy, "A Mechanism for the Solution of an Equation of the nth Degree," this MONTHLY, *1920*, 195–199.

[4] W. H. Bixby, *Graphical Method for Finding the Real Roots of Numerical Equations of Any Degree if Containing but One Variable*, West Point, 1879.

[5] M. E. Lill, "Résolution graphique des équations numériques d'un degré quelconque à une inconnue," *Nouvelles Annales de Mathématiques*, series 2, vol. 6, 1867, p. 359.

[6] L. Cremona, *Graphical Statics*, translated by T. H. Beare, Oxford, 1890, pp. 70–76. [More recent accounts of the method are found in C. Runge, *Graphical Methods* (Columbia University Lectures, 1909–1910), New York, 1912, pp. 11–12, and *Praxis der Gleichungen*, Berlin and Leipzig, 1921, pp. 101–110. See also this MONTHLY, *1911*, 159–162—EDITOR.]

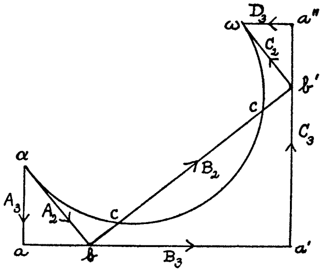
The following extract from the author's 1879 pamphlet will explain the method under discussion.[1] The proof may be easily supplied by the reader. The figure appended is for the case $n = 3$.

"Suppose any numerical equation

$$A_n x^n + B_n x^{n-1} + C_n x^{n-2} + \cdots + T_n x + U_n = 0,$$

in which A_n is any positive number, either whole or fractional.

"Commence, on a blank sheet of paper, at any assumed point α, and using any convenient scale, lay off in a downward direction a distance αa equal to A_n; through a, draw a perpendicular, and lay off upon it with the same scale as before, the value of B_n (laying off this distance to the right if B_n is positive, to the left, if B_n is negative); through the end of B_n, draw a perpendicular to B_n, upon which lay off the value of C_n, upward if positive, downward if negative; and so on. . . . [For each new line the positive direction turns through a right angle counter-clockwise.] Letter the end of the last line ω. We will then have a rectangular contour (that is, a broken line all of whose angles are right angles) of $n + 1$ sides commencing at α and ending at ω.



"Now starting again at α, draw at random any straight line cutting B_n in some point as b; through b draw a perpendicular to αb cutting C_n in some point as b', and so on; the result will be a new rectangular contour of n sides. If the nth side passes through the point ω, then $ab/\alpha a$, taken with its sign changed, is a root of the given equation. There will be as many such contours of n sides, and therefore as many points b, as there are real roots to the given equation.

"These rectangular contours can be readily determined by the aid of a ruler and right-angled triangle.

"Suppose only one root is found by the above method, giving us one new rectangular contour, $\alpha bb' \cdots \omega$, of n sides; call its first side A_{n-1}, its second B_{n-1}, and so on; this new rectangular contour, $\alpha bb' \cdots \omega = A_{n-1}, B_{n-1}$, etc., is the contour which represents the equation obtained by reducing the given equation to the $n - 1$ degree, by dividing out the root, $x = ab/\alpha a$, already found; treat this new contour of n sides like the preceding, obtaining a new rectangular contour of $n - 1$ sides whose first vertex is at some point c upon the line bb'; then $bc/\alpha b$, taken with its sign changed, will be another root of the given equation; and so on.

"The graphic solution of equations, by the above method of trial, will be an exceedingly simple process if the reader will proceed as follows:

"Procure a sheet of paper ruled in squares, each square being about a tenth of an inch on its side (such paper is easy to obtain, as it is nowadays in common use among surveyors and draughtsmen); procure also a sheet of tracing paper or cloth. Lay the tracing sheet upon the ruled paper, and, by the aid of the divisions upon the ruled paper, lay off on the tracing sheet the lines $ABCD$, etc.; thrust a needle, or pin, through the beginning of the A line and also through any convenient intersection upon the ruled paper; then, turning the tracing sheet about the needle as a pivot, the reader can readily follow by eye the ruled lines of the lower sheet, and can easily detect the auxiliary rectangular contours without having to pencil even a single unnecessary line.

"This graphical solution is especially applicable to cases where the desired roots lie between $\pm \frac{1}{2}$ and ± 5; if the desired roots lie beyond these limits, the given equation may advantageously be transformed into another whose roots will lie between the above limits."

In the case of a quadratic equation (represented by the inner contour) the two solutions, if real, may be found by means of a circle on $\alpha\omega$ as diameter, as indicated in the figure.[1]

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[1] For the extension of the method to complex roots and complex equations, see the article "d'après M. Lill" "Résolution graphique des équations algébriques qui ont des racines imaginaires," *Nouvelles Annales de Mathématiques*, series 2, vol. 7, 1868, p. 363—EDITOR.

II. Supplementary Note on the Irrationality of Certain Trigonometric Functions.

By R. S. Underwood, Alabama Polytechnic Institute.

Theorem: *When an angle is rationally expressible in degrees and not a multiple of 30° or 45°, its trigonometric functions are irrational.*

The tangent of any rationally expressible angle except one of the form treated below may be proved irrational by a method so closely analogous to that used in my former paper that the details are omitted here. There remain angles of the form $(m/n)^\circ$, where m/n is an irreducible fraction and where $m = k45$.

In the expansion

$$\tan n\theta = \frac{n \tan\theta - \frac{n(n-1)(n-2)}{3!}\tan^3\theta + \cdots \pm \tan^n\theta (\text{or} \pm n \tan^{n-1}\theta, \text{ when } n \text{ is even})}{1 - \frac{n(n-1)}{2!}\tan^2\theta + \cdots \pm n\tan^{n-1}\theta \text{ (or } \tan^n\theta, \text{ when } n \text{ is even)}}$$

the substitution $\tan n\theta = \pm 1$ gives an nth degree equation whose only possible rational roots (± 1) are excluded by the conditions of the problem. When $\tan n\theta = 0$ or $\pm\infty$, it is evident that the roots obtained by equating the numerator and denominator respectively to zero are either integers or the reciprocals of integers. Furthermore, if $\tan\theta$ is rational, $\tan 2\theta$, when it is finite, is rational. But if $\tan\theta = k$ or $1/k$, k being an integer, $\tan 2\theta = \pm 2k/(1 - k^2)$, which is neither an integer nor the reciprocal of an integer, since the denominator lacks the factor k. Hence $\tan\theta$ is irrational.

The above theorem is interesting in that it brings out in a striking way for pedagogical purposes the great preponderance of irrational over rational numbers. Divide a right angle into n equal parts. Then there are only three distinct angles in all this infinite family which possess rational trigonometric functions.

Incidentally I have shown that the algebraic equations of the form (n odd)

$$x^{n-1} - \frac{n(n-1)}{2!}x^{n-3} + \cdots \pm n = 0,$$

and

$$nx^{n-1} - \frac{n(n-1)(n-2)}{3!}x^{n-3} + \cdots \pm 1 = 0,$$

have no rational roots.

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[1] General Bixby would be glad to hear from any engineer, physicist or mathematician who is frequently required to solve higher equations, what method he uses, how much time is required, and what accuracy is deemed necessary.